Université d'Ottawa · University of Ottawa
Faculté des sciences
Mathématiques et de statistique
Faculty of Science
Mathematics and Statistics

Calculus III for Engineers
MAT 2322A - Fall 2014
Final Exam
Professor: Victor G. LeBlanc
Time limit: 3 hours. Closed books.

Name: Solutions ID Number: __________

Instructions

- This is a closed book exam. Furthermore, all cell phones, pagers or any other electronic or communication devices are forbidden. **The only calculators which are allowed are Texas Instruments TI-30, TI-34, Casio fx-260 and fx-300, scientific and non programmable.**
- This exam has 15 pages and you have 3 hours to complete it.
- Read each question carefully before answering.
- Questions 1 to 10 are multiple choice questions. These questions are worth 2 points each and no partial marks are possible. **Please write your answers in the corresponding boxes in the grid below entitled "Answers to multiple choice Qs".**
- Questions 11 to 16 are long answer questions and are worth 5 marks each, so organize your time accordingly. **A correct answer requires a full, clearly-written and detailed solution.** Answer each question in the space provided, using backs of pages or the extra pages at the end if necessary.
- Do not unstaple the test. Good luck!

Answers to multiple choice Qs

1	2	3	4	5	6	7	8	9	10
D	A	B	A	B	E	D	A	E	C

Grid below is used for grading (do not write in this grid)

MCQ	11	12	13	14	15	16	Total
/20	/5	/5	/5	/5	/5	/5	/50

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1. Consider the function $f(x,y) = x^3y + 2x^2y + x - y$. Which of the following expressions corresponds to the tangent plane to the graph of $z = f(x,y)$ at the point $(x,y,z) = (1,2,f(1,2))$?

A. $z = 15x + 2y$

B. $z = (3x^2y + 4xy + 1)(x-1) + (x^3 + 2x^2 - 1)(y-2) + 5$

C. $z = 5$

D. $z = 15(x-1) + 2(y-2) + 5$

E. $z = 15\vec{i} + 2\vec{j}$

F. This function is not differentiable at the indicated point, so the tangent plane does not exist.

$f_x = 3x^2y + 4xy + 1$ $f_y = x^3 + 2x^2 - 1$

$f_x(1,2) = 15$ $f_y(1,2) = 2$

$f(1,2) = 5$

2. Consider the parametrized curve

$$\vec{r}(t) = \frac{1}{2}t^2\,\vec{i} + \frac{2\sqrt{2}}{3}t^{\frac{3}{2}}\,\vec{j} + t\,\vec{k}, \quad 1 \le t \le 2.$$

What is the total arclength of this curve?

A. $\frac{5}{2}$

B. $\frac{3}{2}$

C. 0

D. 2

E. $\frac{19}{3}$

F. π

$\vec{r}\,'(t) = t\vec{i} + \sqrt{2}t^{\frac{1}{2}}\vec{j} + \vec{k}$

$\|\vec{r}\,'(t)\| = \sqrt{t^2+2t+1} = \sqrt{(t+1)^2} = t+1$

$\int_1^2 (t+1)\,dt = 5/2$

3. For the function $f(x,y) = xe^{xy}$, what is the value of the directional derivative of f at the point $(1,0)$ in the direction of the vector $\vec{u} = -\frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$?

A. -1

B. 0

C. 1

D. 2

E. 3

F. 4

$f_x = e^{xy} + xye^{xy}$

$f_y = x^2e^{xy}$

$\vec{\nabla} f(1,0) = \vec{i} + \vec{j}$

$\vec{\nabla} f(1,0) \cdot \vec{u} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$

4. Which of the following vector fields is conservative?

A. $\vec{F}(x,y,z) = x^2\,\vec{i} + e^y\,\vec{j} + \cos(z)\,\vec{k}$

B. $\vec{F}(x,y,z) = (x+y+z)\,\vec{i}$

C. $\vec{F}(x,y,z) = yz\,\vec{i} + xz\,\vec{j} + xyz\,\vec{k}$

D. $\vec{F}(x,y,z) = -z\vec{i} + x\vec{k}$

E. All of the above

F. None of the above

A. is the only one for which

$\vec{\nabla} \times \vec{F} = \vec{0}$

5. Consider the function $f(x,y) = x^2 - 2y + 6x - y^2 + 2$. Which of the following statements about f is true?

A. f has a local maximum at the point $(x,y) = (-3,-1)$

B. f has a saddle point at the point $(x,y) = (-3,-1)$

C. f has a local minimum at the point $(x,y) = (-3,-1)$ and a saddle point at $(x,y) = (0,0)$

D. f has a local maximum at the point $(x,y) = (-3,-1)$ and a saddle point at $(x,y) = (0,0)$

E. f has a saddle point at the point $(x,y) = (-3,-1)$ and a local maximum at $(x,y) = (0,0)$

F. f has a saddle point at the point $(x,y) = (-3,-1)$ and a local maximum at $(x,y) = (0,0)$.

$$\left.\begin{aligned} f_x &= 2x+6 \\ f_y &= -2-2y \end{aligned}\right\} \rightarrow \begin{aligned} x &= -3 \\ y &= -1 \end{aligned}$$

$f_{xx} = 2$

$f_{xy} = 0$

$f_{yy} = -2$

$f_{xx} f_{yy} - f_{xy}^2 = -4 < 0$

saddle

6. Which of the following expressions corresponds to the integral $\int_0^1 \int_{1-x^2}^1 f(x,y)\,dy\,dx$ with order of integration reversed?

A. $\int_1^0 \int_1^{1-x^2} f(x,y)\,dy\,dx$

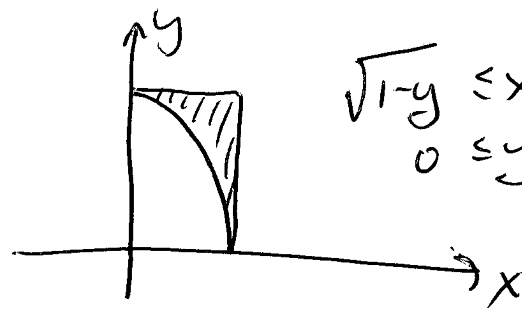
B. $\int_{1-x^2}^1 \int_0^1 f(x,y)\,dx\,dy$

C. $\int_{1-x^2}^1 \int_0^1 f(y,x)\,dx\,dy$

D. $\int_0^1 \int_{1-y^2}^1 f(y,x)\,dx\,dy$

E. $\int_0^1 \int_{\sqrt{1-y}}^1 f(x,y)\,dx\,dy$

F. $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$



$\sqrt{1-y} \le x \le 1$

$0 \le y \le 1$

$$\overbrace{}^{P} \quad \overbrace{}^{Q}$$

7. Consider the vector field $\vec{F}(x,y) = \underbrace{(2x+2y)}_{}\,\vec{i} + (2x+2y)\,\vec{j}$, and let C be the circle $x^2+y^2=4$ oriented counter-clockwise. Then the line integral

$$\int_C \vec{F}\cdot d\vec{r}$$

is equal to

$$\frac{\partial Q}{\partial x} = 2 \qquad \frac{\partial P}{\partial y} = 2$$

$\vec{F}$ is conservative, and C is closed.

A. -3

B. -2

C. -1

D. 0

E. 1

F. 2.

8. Consider the three-dimensional solid in the **first octant** which is bounded between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$, and the cones $z = \frac{\sqrt{3}}{3}\sqrt{x^2+y^2}$ and $z = \sqrt{3}\sqrt{x^2+y^2}$. This solid has a mass-density given by $\delta(x,y,z) = x$. What is the total mass of this solid?

A. $\frac{5\pi}{16}$

B. $\frac{3\sqrt{3}-3}{4}$

C. $\frac{\pi}{8}$

D. 0

E. $\frac{\pi}{12}$

F. None of the above.

$$\int_{\theta=0}^{\pi/2}\int_{\phi=\pi/6}^{\pi/3}\int_{\rho=1}^{2}(\rho\sin\phi\cos\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \cdots = \frac{5\pi}{16}$$

9. Consider the two-dimensional region D drawn below, whose boundary is the oriented curve C, also drawn. Let $\vec{F}(x,y) = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$ be a vector field with continuous partial derivatives. Then which of the following equations corresponds to Green's theorem?

A. $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dx\,dy$

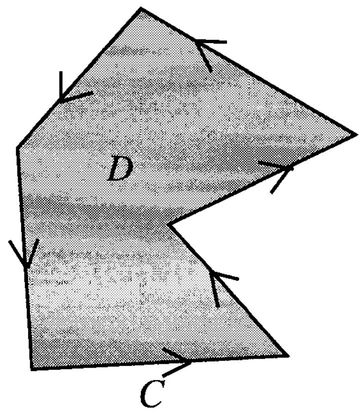
B. $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dx\,dy$

C. $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$

D. $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) dx\,dy$

E. $\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

F. $\int_C Q\,dx + P\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$



See notes

10. Consider the parametrized surface S described by

$$\vec{r}(\theta, z) = 2\cos\theta\,\vec{i} + 2\sin\theta\,\vec{j} + z\,\vec{k}, \quad \frac{\pi}{4} \le \theta \le \frac{\pi}{2}, \; 0 \le z \le 3,$$

and the scalar function $f(x,y,z) = yz$. What is the value of the surface integral $\iint_S f\,dS$?

A. $7\sqrt{2}$

B. $8\sqrt{2}$

C. $9\sqrt{2}$

D. $10\sqrt{2}$

E. $11\sqrt{2}$

F. $12\sqrt{2}$

$\vec{r_\theta} = -2\sin\theta\,\vec{i} + 2\cos\theta\,\vec{j}$

$\vec{r_z} = \vec{k}$ $\quad \|\vec{r_\theta} \times \vec{r_z}\| = 2$

$$\int_{\pi/4}^{\pi/2} \int_0^3 (2\sin\theta)\cdot z \cdot 2\,dz\,d\theta =$$

$$\ldots = 9\sqrt{2}$$

11. Consider the solid defined by the inequalities

$$0 \le z \le 8 - x^2 - y^2,$$
$$0 \le x \le 2,$$
$$0 \le y \le 1.$$

This solid has a mass density given by $\delta(x, y, z) = x + 2y$. Find the total mass of this solid.

$$\text{Mass} = \int_0^1 \int_0^2 \int_0^{8-x^2-y^2} (x+2y)\,dz\,dx\,dy =$$

$$\int_0^1 \int_0^2 (x+2y)z \Big|_0^{8-x^2-y^2} dx\,dy = \int_0^1 \int_0^2 (8x - x^3 - xy^2 + 16y - 2x^2y - 2y^3)\,dx\,dy$$

$$= \int_0^1 \left(4x^2 - \frac{x^4}{4} - \frac{x^2y^2}{2} + 16xy - \frac{2x^3}{3}y - 2xy^3 \Big|_0^2 \right) dy =$$

$$\int_0^1 \left(16 - 4 - 2y^2 + 32y - \frac{16y}{3} - 4y^3\right) dy =$$

$$12y - \frac{2y^3}{3} + 16y^2 - \frac{8}{3}y^2 - y^4 \Big|_0^1 = \cdots$$

$$\frac{71}{3}$$

12. Find the global extrema for the function $f(x,y) = 2x^2 + 3y^2 - 4x - 5$ on the disk of radius 4, centered at the origin, $0 \le x^2 + y^2 \le 16$.

$f_x = 4x - 4 \qquad f_y = 6y \qquad$ C.P. at $(1,0) \in D$. Keep it.

Boundary

$$\left.\begin{aligned} 4x-4 &= 2\lambda x \\ 6y &= 2\lambda y \\ x^2+y^2 &= 16 \end{aligned}\right| \longrightarrow 2y(3-\lambda)=0$$

$\underline{y=0}$ or $\underline{\lambda=3}$

$y=0$:

$x=4$ or $x=-4$

$16-4 = 8\lambda \qquad -16-4 = 8\lambda$

$\lambda = 12/8 \qquad \lambda = -\frac{20}{8}$

$\boxed{(4,0),\ (-4,0)}$

$\lambda = 3$:

$4x-4 = 6x$

$2x = -4$

$x = -2$

$4 + y^2 = 16$

$y^2 = 12$

$y = \pm\sqrt{12}$

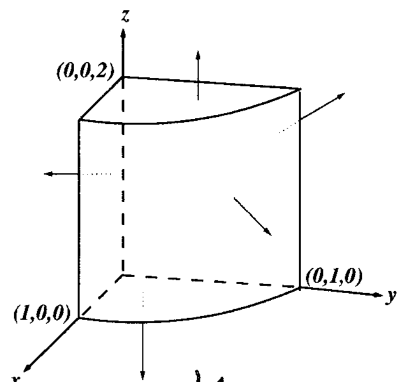
$\boxed{(-2,\sqrt{12}),\ (-2,-\sqrt{12})}$

(x,y)	$f(x,y)$	Global
$(1,0)$	-7	MIN
$(-2,\sqrt{12})$	47	MAX
$(-2,-\sqrt{12})$	47	MAX
$(4,0)$	11	
$(-4,0)$	43	

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

13. For the vector field $\vec{F}(x,y,z) = \underbrace{(x^2y+y\sin z)}_{P}\,\vec{\imath} + \underbrace{(xy^2+ze^x)}_{Q}\,\vec{\jmath} + \underbrace{(xy+y^2)}_{R}\,\vec{k}$, compute the divergence of $\vec{F}$, i.e. compute $\vec{\nabla}\cdot\vec{F}$. Then, **using Gauss' divergence theorem**, compute the surface (flux) integral $\int\int_S \vec{F}\cdot d\vec{S}$, where S is the outward-oriented surface illustrated below. Note that S is the boundary of the solid region D which is the portion of the cylinder $x^2+y^2 \le 1$, $0 \le z \le 2$ which lies in the first octant.

$$\vec{\nabla}\cdot\vec{F} = 2xy + 2xy + 0 = 4xy$$



By Gauss' divergence theorem, we get

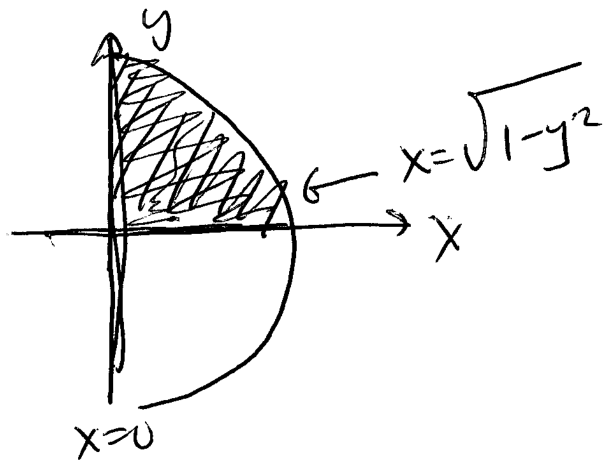
$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_D \vec{\nabla}\cdot\vec{F}\, dV.$$

This triple integral, we will evaluate using cylindrical coordinates

$$= \int_{\theta=0}^{\pi/2}\int_{z=0}^{2}\int_{r=0}^{1} 4(r\cos\theta)(r\sin\theta)\cdot \underbrace{r\,dr\,dz\,d\theta}_{dV}$$

$$= \cdots = 1$$

14. Consider the integral $\int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} e^{x^2+y^2}\,dxdy$. Convert it to polar coordinates, and then evaluate the integral.



In polar coordinates, this region is described as $0 \le r \le 1$

$0 \le \theta \le \pi/2$

So the integral becomes

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{1} e^{r^2} \cdot \underbrace{r\,dr\,d\theta}_{dA} = \int_{\theta=0}^{\pi/2} \left(\tfrac{1}{2} e^{r^2} \Big|_0^1 \right) d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left(\tfrac{1}{2} e^1 - \tfrac{1}{2} e^0 \right) d\theta = \left(\tfrac{1}{2} e - \tfrac{1}{2} \right) \cdot \pi/2$$

$$= \pi/4\,(e-1)$$

15. Consider the vector field $\vec{F}(x,y) = \overbrace{(2x + y\sin\, y)}^{P}\,\vec{\imath} + \overbrace{(x\sin\, y + xy\cos\, y)}^{Q}\,\vec{\jmath}$.

(a) Show that the vector field $\vec{F}$ is conservative.

(b) Find a potential for $\vec{F}$, i.e. find a scalar function $f(x,y)$ such that $\vec{F}(x,y) = \vec{\nabla} f(x,y)$

(c) Compute the line integral $\int_C \vec{F}\cdot d\vec{r}$, where C is the oriented straight line segment that starts at $(x,y) = (0,0)$ and ends at $(x,y) = (1,\pi/2)$.

a) $\frac{\partial Q}{\partial x} = \sin y + y\cos y = \frac{\partial P}{\partial y} = \sin y + y\cos y$ on $\mathbb{R}^2$

So $\vec{F}$ is Conservative

b) $\frac{\partial f}{\partial x} = 2x + y\sin y \Rightarrow f(x,y) = x^2 + xy\sin y + h(y)$

$\frac{\partial}{\partial y} \Rightarrow x\sin y + xy\cos y + h'(y) = Q$

$= x\sin y + xy\cos y$

$\Rightarrow h'(y) = 0 \Rightarrow h = C$

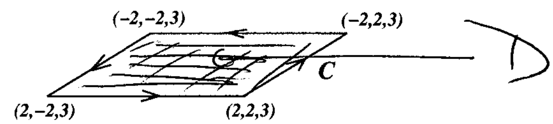
If we take $C=0$, then a potential is

$$\boxed{f(x,y) = x^2 + xy\sin y}$$

c) $\int_C \vec{F}\cdot d\vec{r} = f(1, \pi/2) - f(0,0)$

$= 1 + \pi/2 \sin \pi/2 - 0 = 1 + \pi/2$

16. Consider the vector field $\vec{F}(x,y,z) = (-yz + xe^{x^2})\,\vec{\imath} + (xz + y\sin(y^2))\,\vec{\jmath} + (e^z + z\cos(z))\,\vec{k}$, and let C be the oriented curve illustrated below consisting of the four straight line segments which run from $(2,-2,3)$ to $(2,2,3)$ to $(-2,2,3)$ to $(-2,-2,3)$ and then back to $(2,-2,3)$. Compute the line integral $\int_C \vec{F}\cdot d\vec{r}$ using either a direct computation, or using Stokes' theorem. Note that one of these two methods will yield a much simpler computation than the other one, so choose the method carefully.



$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -yz+xe^{x^2} & xz+y\sin y^2 & e^z + z\cos z \end{vmatrix} = \vec{\imath}(-x) - \vec{\jmath}(y) + \vec{k}(z+z)$$

$$= -x\vec{\imath} - y\vec{\jmath} + 2z\vec{k}$$

C is the oriented boundary of the square D, parametrized as

$$\vec{r}(x,y) = x\vec{\imath} + y\vec{\jmath} + 3\vec{k} \qquad -2 \le x \le 2, \quad -2 \le y \le 2$$

$$\vec{r}_x = \vec{\imath}, \quad \vec{r}_y = \vec{\jmath}, \quad \vec{r}_x \times \vec{r}_y = \vec{k}$$

By Stokes' theorem, we have $\int_C \vec{F}\cdot d\vec{r} = \iint_D (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$

$$= \int_{-2}^{2}\int_{-2}^{2} (-x\vec{\imath} - y\vec{\jmath} + 2\cdot 3\vec{k})\cdot(\vec{k})\,dx\,dy = \int_{-2}^{2}\int_{-2}^{2} 6\,dx\,dy$$

$$= 6\cdot 4\cdot 4 = 96$$